Good Questions in Linear Algebra

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Contents

Chapter 1

Introduction to the use of good questions

This book contains questions about linear algebra, intended for class discussion. It also contains comments about the questions, because the real purpose of the book is as an example of how to create good questions.

The material in this book was inspired by the GoodQuestions Project at Cornell University, which was supported by the National Science Foundation grant DUE-0231154. The questions here were developed over a period of several years, and used in large lectures (175-225) to students of engineering and science. I have found that good questions help me understand what parts of the subject cause difficulty for the students. A good question also gives students a way to consolidate their understanding by means of discussion with their peers. This collection does not include all topics of a linear algebra course. Far from it; the book is an example of how to create good questions.

The questions are displayed in large print for projection. For those who wish to create digital or print versions of their own questions, an example of LaTeX source is given.

Instructors may use questions in various ways. I use them mainly to stimulate class discussion of topics that have been introduced one or two lectures previously, that is, topics that the students are somewhat familiar with.

These questions fall into approximately four types, or uses:

(a) Some questions are quick checks on mostly factual knowledge, serving as a reminder to the students of some prerequisite needed for the next topic of the lecture. The topic might be something recently done in this course, or review something from a previous one.

(b) Some questions create discussion among the students, usually by offering a small number of alternate statements from which to choose. Among the statements there is always at least one correct, and usually one which contains some correct idea but is slightly ambiguous.

(c) Other questions allow the creation, in the mind of each individual student, of some mathematical fact or link between facts. These do not necessarily depend on peer discussion. These questions, when they work as intended, give possibly the truest experience of the mathematician's "ah-ha!"

(d) Finally there are questions which punctuate the lecture, by which I mean the following. Most of the class time consists of lecture on the blackboard in a lighted room. For a few minutes in the middle of the lecture the room goes dark, the projector comes on, and there is something new to think about. That change of setting is sufficient to draw the students' attention to whatever is displayed. A few questions use this special time to introduce new ideas, or make subtle or important points that are otherwise missed.

I introduce a question in a large lecture class by first showing it from an overhead projector and reading a small part of it to the students, then saying to the class, "Now I'll be quiet so you can read it." I give them about 30 seconds to read it, then we vote by a show of hands: "Who thinks item 1 is true? 2? 3?" Then perhaps "Uh-oh! Some of you didn't vote!" What happens next depends on the results of the vote. In some cases there is controversy and disagreement, which usually is good. In that case, I say "You better discuss that with the friendly person sitting next to you." Then I listen as they discuss it for a short time. We then take another vote. The best outcome is that now most people know the right answers. I can say to the class in astonishment, "Wow! you guys can teach each other." They like that, and it is one of the best results possible. I move on to the next topic.

In some cases there is a strong correct first vote and discussion isn't necessary. In other cases the vote is mixed even after the discussion. This last case is a warning to the instructor that the question was too hard, or ambiguous, or otherwise inappropriate. I can suggest to the class that they think it over again as part of their study, and I also attempt to improve the question for its next use.

This process is essential to the creation of good questions. The instructor simply cannot anticipate what will be a good question. The role of the class in voting, discussing, and revoting is essential.

Chapter 2

Early concepts of linear equations

In the beginning of any new study, there are many things for students to sort out: new terminology, new uses of familiar language, and necessary prerequisites. In this chapter are examples of questions which can be used on the first day to reveal common misunderstandings.

Many books on linear algebra begin by solving linear equations. In a mathematics book we might write a linear equation as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

but they don't always look like that in applications. So we ask a question about linear equations.

This is a successful early question, one of the first given, while the students are just beginning to find out what the class is going to be like. Students like it because it allows them to meet new people and talk about something useful and not too hard, and because it clarifies that our abstract linear equation includes many examples which look somewhat different. For example, here is a typical equation from an applied text:

$$q + \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} + W = u_2 - u_1 + \frac{V_2^2 - V_1^2}{2g} + z_2 - z_1$$

The students are faced with many such equations every day.

$$xy + 6uv - \pi zw = 8t - 22$$

Which are true?

(a) If the variables are y,v,w, and t (and x, u, π, and z are constants) then the equation is linear.

(b) If the variables are just x and z, and all other symbols are constants, then the equation is linear.

(c) If x and y are variables then the equation is not linear.

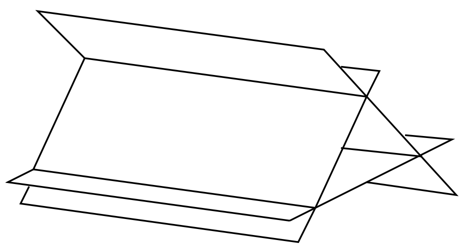
The intersection of several planes consists of the set of points which satisfy all their equations. Thus, “intersection” is a central concept when dealing with a system of equations.

Part (a) is a check on the students’ perceptions. “Perception” is not a mathematical concept, and varies from one person to the next, even when no optical illusion is present or intended. The word “It” is not specified, but presumably refers to the whole figure.

Parts (b) and (c) are about the word “intersection.” Usually part (c) creates some discussion, as intended, and it is necessary for me to clarify “intersection”.

Part (d) doesn’t refer to the figure, and it could be used as a separate question. In fact this whole question could be subdivided into three, but class time is saved by combining some items.

Initially there will be some votes in favor of each of these items. Item (c) is true. Item (d) is false, and one should probably say that item (b) is also false, though it is too vague to be considered a mathematical statement.



What do you think?

(a) It looks 3-dimensional.

(b) It shows 3 planes intersecting in 3 lines.

(c) It shows that 3 planes do not have to have any points in their intersection.

(d) Two planes certainly intersect in a line.

This is a relatively easy question, but very worthwhile because there are always students who say, initially, that the answer is two. These students learn quickly from their peers that the answer is one.

Note that unlike many of these questions, there is no way to reason this one out. It is a question of knowing what is meant by “solution” to a system of equations.

In most cases, a question which simply asks for a fact is probably not going to qualify as a “good” question. In this case, I have found that the discussion was pleasing and informative to the students, and on this ground it qualifies as good.

Solving a system

$$ax_1 + bx_2 = 5$$

$$cx_1 + dx_2 = 6$$

suppose we find that

$$x_1 = 3$$

$$x_2 = 5$$

Is that one solution or two?

These questions are much harder than they look.

A natural approach to the first question for many students is to solve $5x = 3$ as $x = 3/5$, then solve $40x = 24$ for $x = 24/40 = 3/5$, then answer yes. It requires greater maturity to do literally what it says: If $5x = 3$, then $8(5x) = 8(3)$, answer yes. What is more "mature" about that? You need to discuss and manipulate an equation without solving it. Students are often not prepared to do this.

The second question is a variation of the first, for the homogeneous case.

Later in linear algebra we point out that for homogeneous equations, any scalar multiple of a solution is again a solution. This is of course different from the fact that any equation may be multiplied by an invertible number without changing the solutions. The question helps students think about the distinction between the two statements.

Consider the simple arithmetic problem

$$5x = 3.$$

If you multiply the equation by 8, do you get the same answer to the new equation?

Consider the equation and some solutions

$$x - 2y = 0, \qquad (x, y) = (0, 0),\ (2, 1),\ (4, 2).$$

(a) If you multiply the equation by 8, do you get the same answers to the new equation?

(b) If you multiply the answers by 8, do you get more answers to the original equation?

This question is intentionally ambiguous. It does not specify how many variables there are. By this means we hope to stimulate discussion. Item (b) is there to suggest that one needs to be clear about what the variables are.

If the variables are x_1 and x_2 then item (a) is correct, but if there are three or more variables than item (c) is correct.

Consider the system of equations

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ 0 &= 0. \end{aligned}$$

Which statements apply?

(a) There is one answer.

(b) There is no requirement about x_3.

(c) There are infinitely many solutions.

Chapter 3

Solving systems of linear equations

Many questions in this chapter are about row operations.

This question is good for checking students' computational skill, and it is not difficult. I don't take an initial vote on a question of this type. Just allow students to work on it together for a few seconds, then vote once.

$$\begin{bmatrix} 1 & 7 & 0 & 3 & 5 \\ 1 & 0 & 1 & 0 & 16 \\ 2 & 14 & 0 & 0 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & 0 & 3 & 5 \\ 2 & 0 & 2 & 0 & 32 \\ 0 & 0 & 0 & -6 & 0 \end{bmatrix}$$

Which row operation was not done?

a) (row 2) $\to$ 2(row 2)

b) interchange (row 2), (row 3)

c) (row 3) $\to$ (row 3) $-$ 2(row 1)

This might be a quick check, or it might require some work, depending on the state of the class. The choices provided are so limited that students can be encouraged to try a small example to resolve it.

The question leads into a discussion about the reversibility of row operations.

Suppose a matrix A is transformed to matrix B by the row operation:
(row 2) $\rightarrow$ (row 2) $-$ 3(row 5)
Which operation transforms B back to A?

(a) (row 2) $\rightarrow$ (row 2) $- \frac{1}{3}$(row 5)

(b) (row 2) $\rightarrow$ (row 2) $+$ 3(row 5)

(c) You can't do it.

I use this question is a quick check on the student's knowledge. Since I have advised students to read ahead, I can use the question on the day I give the definition of the row-echelon form, and most students will get this correct the first time. During the discussion, those who don't know will learn from others seated near them what the answers are, and they will be encouraged to read ahead too. After they discuss the question, I usually ask what the problem is with the 4×5 matrix, and they always know it is the 13. Sometimes they don't know that the 4×1 needs another row interchange.

Which of these matrices are in reduced row–echelon form?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 17 & 0 & 13 & 5 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This question usually starts some intense discussion. It might be helpful to suggest that they write examples of each possibility.

Since one looks for leading 1's in each row, there certainly can't be 4 of them. But this won't be obvious until one writes out an example. The other extreme case, 0, can occur but only for one particular matrix. The question helps students practice thinking about the extreme and special cases of things.

How about the number of leading 1's in the reduced row–echelon form of a 3×4 matrix?

(a) There must be 3 of them.

(b) There may be 0, 1, 2, or 3.

(c) There could be 4.

It is an important point about the matrix–vector product that this question can be answered without calculation; $\vec{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Since the students are encouraged to work together on it, and many will know this fact, all will come away from the problem appreciating it.

The question reminds students that the matrix–vector product $A\vec{x}$ is a linear combination of columns of A.

Using that

$$2\begin{bmatrix}1\\-1\end{bmatrix} - 3\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}-4\\-17\end{bmatrix}$$

(or not), find a solution to

$$\begin{bmatrix}1 & 2\\-1 & 5\end{bmatrix}\vec{x} = \begin{bmatrix}-4\\-17\end{bmatrix}$$

.

This question reveals many misconceptions!

Students might not be able to parse the first sentence simply because the "variable" is different in the two equations. But there is not really a variable, just two specific solutions whose names are "x" and "y".

Some students do not understand that the question is whether $A(x+y) = b$ or not. Some ask for clarification about whether the "b" is the same in both equations. They will not, for some reason, question whether "A" is the same.

Some students add the two equations to find that $A(x + y) = 2b$, and then incorrectly answer True because the equation has the same general form.

Some students comment that it is True if $\vec{b}$ is zero. The question connects with the fact that the null space of a matrix is closed under addition, but can be used whether or not this is already known to the students.

I have not tried writing $\vec{x}_1$ and $\vec{x}_2$ instead of $\vec{x}$ and $\vec{y}$; it might be worth trying.

Suppose we found two solutions to the same equation: $Ax = b$, and $Ay = b$.
True–False: we can get another solution to the same equation by just adding: $x + y$ is also a solution.

This question seems to be harder than average. Items (a) and (c) are correct. On the initial vote there are typically some students in favor of item (b). The contrast between items (b) and (c) makes a useful discussion among the students. They teach each other.

Fundamentally this question is only about linearity in the form $A(\vec{x} \pm \vec{y}) = A\vec{x} \pm A\vec{y}$, and how that relates to solutions of homogeneous and inhomogeneous equations. Part (a) also bears on terminolgy that some of the students have seen in differential equations courses, about "adding homogeneous and particular" solutions. I've found that some students ask about this even though it is not explicit in the question.

It is somewhat characteristic of my questions that one item, such as (c) here, may be used to focus the students' attention and peer discussion on another item, here (b).

Suppose that $A\vec{u} = \vec{b}$ and $A\vec{z} = \vec{0}$. Which are true?

(a) $A(\vec{u} + \vec{z}) = \vec{b}$

(b) $A(\vec{u} - \vec{v}) = \vec{0}$ and therefore $\vec{u} - \vec{v} = \vec{z}$

(c) $A(\vec{u} - \vec{v}) = \vec{0}$

The first question shown here is difficult for students. To see that item (a) is correct they need to calculate symbolically, $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{b} + \vec{0}$. Some students seem to have little trouble with this, but others are puzzled even after peer discussion.

Part of the problem here may be that many words in the language of linear algebra need to be followed by a phrase "of (...)", where the dots represent another word of the language. In this case, you don't want the student to think of $\vec{x}$, $\vec{y}$ and $\vec{x}+\vec{y}$ merely as solutions, but rather as solutions of specified equations.

Recognizing this situation, I made the second question to give practice with the language. Students seem to enjoy discussing it.

Suppose we know solutions of the two equations: $A\vec{x} = \vec{b}$, and $A\vec{y} = \vec{0}$.
Which is correct?

(a) We can get another solution of the first equation by just adding: $\vec{x} + \vec{y}$ is also a solution.

(b) $\vec{x} + \vec{y}$ is a solution of the second equation.

Match X of Y:

X	Y
bunch	transformation
domain	vector
coordinates	flowers
linear dependence	matrix
matrix	subspace
columns	list of vectors
basis	linear space
linear combination	basis
solution	$\mathbb{R}^n$
subspace	equation

This is a quick check of basic facts about the inverse. The answer is (b) because $\vec{x} = A^{-1}\vec{b}$. I write this formula on the blackboard after the class discussion and voting are finished. Since the formula is not written into the text of the problem, students have a chance to come up with it by themselves or in peer discussion.

If A is an invertible matrix, how many solutions does any system $A\vec{x} = \vec{b}$ have?

(a) 0

(b) 1

(c) ∞

This question was initially an experiment. I thought it was not very well designed, but the students liked it as a check on basic algebra.

Items (a) and (c) are correct, while (b) is meaningless if A is a matrix, but correct if A is a nonzero number. In fact, no student has ever asked whether A might represent a number.

Assume that $A\vec{x} + \vec{b} = \vec{c}$. Which are true?

(a) $A\vec{x} = \vec{c} - \vec{b}$

(b) $\vec{x} = (\vec{c} - \vec{b})/A$

(c) If A^{-1} exists then $\vec{x} = A^{-1}(\vec{c} - \vec{b})$.

This question may be used after some discussion of standard basis vectors, and inverse. If students check item (a) by calculation, then (b) follows.

The question assumes that the notation $\vec{e}_2$ is being used for the second standard basis vector.

Part (c) is difficult if the students don't yet know that for any matrix B, the second column of B is $B\vec{e}_2$, particularly since here one has to think about $A^{-1}\vec{e}_2$ as a linear combination of columns of a matrix which is not shown.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 6 \\ -2 & 1 & 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{6} \end{bmatrix}, \quad A^{-1} \text{ exists.}$$

(a) True-False: $A\vec{v} = \vec{e}_2$

(b) True-False: $A^{-1}\vec{e}_2 = \vec{v}$

(c) What is column 2 of A^{-1}?

Chapter 4

Vector spaces

where does this go?

Discussions of the null space and column space of a matrix tend to treat the matrix as the central concept. It comes as a revelation to the students that the same subspace may be described as the null space of A and the column space of B, and neither A nor B is uniquely determined. In fact, the usual algorithm for solving $A\vec{x} = \vec{0}$ produces columns for B.

I had set an exam question which asked whether a certain vector was in the nullspaceof a certain matrix. A few students tried to answer by finding a basis for the nullspace, then express the given vector as a linear combination of the basis vectors. Of course, that will work, but I hope nobody does it that way!

For this reason I wrote this question with a limited menu of options. Still item (a) is not recommended. Item (b) is much faster, and uses only the definition of nullspace.

The question encourages students to know and use the definitions of key words before launching into computations.

I think the students also enjoy knowing something that Joe doesn't.

Joe thinks $\vec{x}$ is in the nullspaceof A, where
$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & 0 \\ 3 & 10 & 5 \end{bmatrix}$. Which are correct?

(a) To check, he should do row operations.

(b) To check, he should multiply $A\vec{x}$.

(c) Joe is wrong.

This question tests the definition of the column space of a matrix. Many students see (a) right away, and that (b) would take more thought. Their reactions suggest that part (c) is surprisingly hard, possibly due to the $\sqrt{2}$.

Which two of these (all are true) can be verified without any calculations?

(a) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is in the column space Col $\begin{bmatrix} 3 & 0 & 0 \\ 3 & 10 & -1 \\ 4 & 20 & -2 \end{bmatrix}$.

(b) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is in Col $\begin{bmatrix} 3 & 0 & 0 \\ 3 & 5 & 6 \\ 4 & 7 & 8 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is in Col $\begin{bmatrix} 3 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & \sqrt{2} \end{bmatrix}$.

This question is about a basis of $\mathbb{R}^2$. Students will know that a basis for a vector space is a list of vectors having two properties, but I didn't use the word "span" because the question already contains plenty of things to think about. That decision is in line with my observation that briefer questions tend to be more successful.

Students had no trouble seeing that the (?) coefficient ought to be $y - x$, or that the two vectors are independent. There was some hesitation about the third part of the question, but enough people said "$\mathbb{R}^2$" for the others to hear, and agree.

- $\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (?) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly (dependent, independent).
- $\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$ is a basis for ________.

This question concerns the idea of a subspace of $\mathbb{R}^4$.

Before the question is used, students will know that a subspace of $\mathbb{R}^4$ must contain $\vec{0}$ and be closed under addition and scalar multiplication.

The question illustrates a set of vectors for which only two of the three requirements are met. To keep the question concise, I did not ask whether $\vec{0}$ is in X, but I always mention this to the class after they discuss it.

Let X be the set of all vectors $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in $\mathbb{R}^4$ having $ad = bc$. True–False:

(a) X is closed under scalar multiplication.

(b) X is a subspace of $\mathbb{R}^4$.

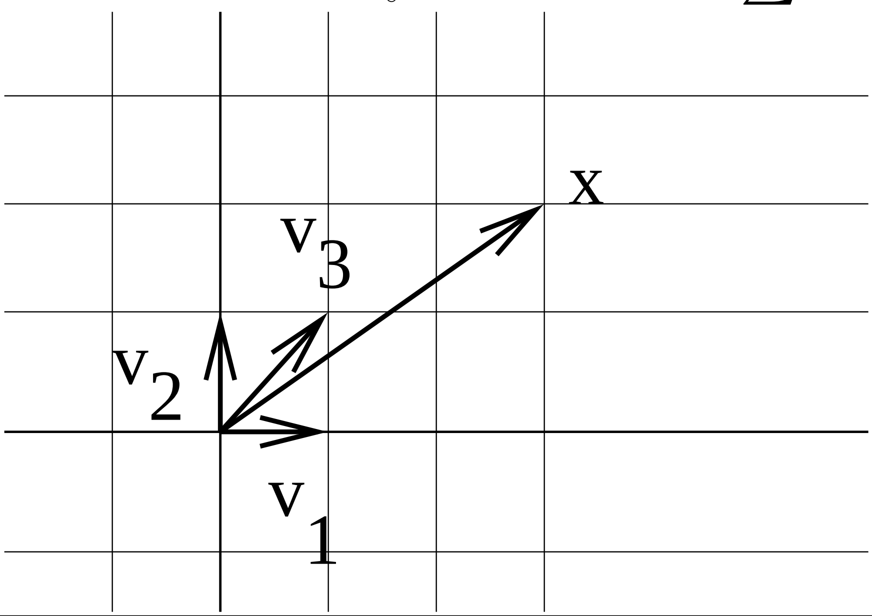
This question can be used to reinforce the idea of linear combination. Several students have challenged themselves to find combinations of the form

$$\vec{x} = ()\vec{v}_1 + ()\vec{v}_2 - \frac{1}{2}\vec{v}_3.$$

It can also be used in connection with linear independence, since $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ are dependent.

If used after introducing bases, the question can highlight the uniqueness of coordinates relative to a basis.

Sketch several ways to write $\vec{x} = \sum c_k \vec{v}_k$.



Unlike most words in linear algebra, “span” is both a noun and a verb. This question addresses the noun. Items (b) and (c) are correct. Some students initially vote in favor of (a), and are quickly corrected by their peers.

Which are correct? The number of vectors in the

$$\text{span}\{\vec{u}, \vec{v}, \vec{w}\}$$

(a) is 3

(b) is ∞ or 1

(c) cannot be determined unless you know $\vec{u}$, $\vec{v}$, and $\vec{w}$.

I wrote this question after grading a problem on a test:

"Vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent when ________________________."

Several students gave item (a) as their response, which seemed puzzling to me at first. So I wrote the question to find out about it.

It turns out that the students were programmers, and that computer languages like C evaluate "P and Q" by first testing P, stop if P is false, and if P is true then test Q. For this reason they felt that statements (a) and (b) are equivalent.

Do these statements convey the same information?

a) $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = 0$ and all $c_k = 0$

b) if $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = 0$ then all $c_k = 0$

This question has to do with thoughtful reading of mathematics, and with organizing one's knowledge. It points out that even when two statements are true and superficially similar, the status of the statements might be entirely different.

Students have given mixed reactions; some are enlightened, some puzzled, but most sort it out after some discussion.

Consider two statements about a 5 by 7 matrix A.

(a) The columns of A span its column space.

(b) The columns of A span $\mathbb{R}^5$.

One of them is a definition, and the other one contains information about this specific matrix. Which is which?

Students seem to enjoy assigning credit to the fictitious Joe. He appears in several questions, usually making an assertion that I actually found on a student's paper or in conversation.

On this question, all three answers receive enthusiastic votes from the class. I tell them that (c) is the best answer because Joe has some important facts about the column space, but he doesn't understand how many vectors there are in it. The students accept this without enthusiasm, perhaps because they can identify somewhat with Joe.

On a recent test, a student wrote "the column space is the amount of space filled up by taking all the columns and all their possible linear combinations." I like the way she expanded the packed language of linear algebra into a more human one.

Joe wrote on his paper: "the column space of a matrix is the pivot columns". He ought to receive

(a) full credit

(b) no credit

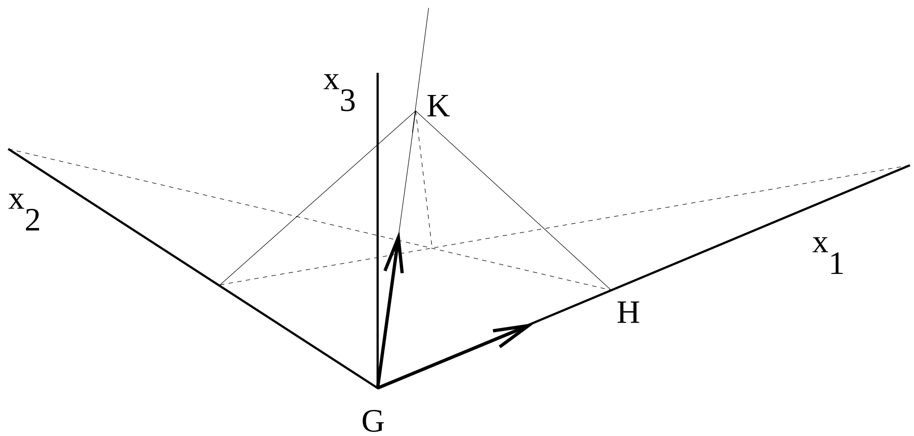
(c) half credit

This question is about coordinates relative to a basis. I thought it was too complicated and too ambiguous. The class thought it was interesting, so I kept it.

Item (1) contains a reminder that not every plane is a subspace.

Item (2) points out that coordinates are determined only after a basis is selected.

Most students choose (c) but some also point out that (a) and (b) seem to correspond correctly with the two given bases. That is, they seem to have understood the question exactly as I intended it.



The pyramid designer wants to use (x_1, x_2, x_3) coordinates in the $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ basis. The guy in charge of the right face wants to use the two bold vectors as a basis.

1. Are the bold vectors a basis of the plane containing the right face GHK? (Is that plane even a subspace of $\mathbb{R}^3$?)

2. Which are likely the coordinates of the top point K? a) $(0, 2)$ b) $(1, 1, 1.4)$ c) too ambiguous to say

This question has to do with composing linear transformations. They usually have seen this rotation matrix already.

I've found that the question starts a considerable discussion among the students, so I often omit taking an initial vote, but instead let them read it and discuss it.

WIthin a minute or two they usually agree that (a) is True and that $\theta = 70^o$ in (b).

$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ gives rotation of $\mathbb{R}^2$ by angle θ.

(a) True–False: $R_{30^o} R_{40^o} = R_{40^o} R_{30^o}$

(b) $R_{30^o} R_{40^o} = R_\theta$ for some θ. What is θ?

Some linear algebra courses will discuss vector spaces other than subspaces of Euclidean spaces.

These questions introduce the language of linear algebra into spaces other than $\mathbb{R}^n$.

In most cases, my questions do not preview or introduce material. They are used typically one or two lectures after the topic of the question is first covered. These two questions were used differently, as you can tell by the mention of a new context. Students seemed interested in the new contexts.

I've found that projection to the screen punctuates the lecture if it is used sparingly. I use the projector for only a few minutes near the middle of most lectures. For these questions, the break and change in lighting signal the new use of the language.

We don't vote on these.

Would you say that the functions

$$f(x) = 3 - x^2, \quad g(x) = 4 + x$$

are linearly independent?
Note that this question uses familiar words in a new context.

Would you say that the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are a basis for the set of all 2 by 2 matrices?
Note that this question uses familiar words in a new context.

Here we ask about the nullspaceof a linear transformation that they know from calculus or from a differential equations course.

Items (a) and (b) are intended to help students break away from matrices. Most students agree that (a) is True and (b) False, after a brief discussion.

Item (c) is much harder, and is there to pave the way for the next part of the lecture. That lecture contains an example in which the domain of T consists of polynomials, and a second example in which the domain is the span of $\cos t$ and $\sin t$. The nullspaceis very different in the two cases, which is a genuinely new idea to students at this stage.

$T(f) = f''$. To find the nullspaceof T, what do I need to do?

(a) solve the differential equation $f'' = 0$.

(b) figure out a matrix for T.

(c) know what the domain of T is.

Chapter 5

Orthogonality

The ideas of orthogonality and projections to a subspace are challenging concepts.

This question concerns the projection of a point to a line in $\mathbb{R}^2$.

At this stage, the students have usually done this both in multivariable calculus and physics courses. In physics it is usually called the component of $\vec{x}$ in the direction of $\vec{u}$. With this experience, students usually can figure out that the formula is incorrect because $\vec{u}$ is not a unit vector; the correct answer is

$$\frac{\vec{u} \cdot \vec{x}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{7}{25} \vec{u}.$$

The question is important because it is the prototype of projections onto subspaces of $\mathbb{R}^n$.

Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$. True–False: the closest point to $\vec{x}$, on the line spanned by $\vec{u}$, is given by

$$(\vec{u} \cdot \vec{x})\vec{u} = 7\vec{u}.$$

If False, how can you fix it?

A student's inquiry about the size versus rank of a matrix suggested this question.

On the first vote, the class gave the most vote to (1a), (1b), and (2b). After they talked it over bridfly, the (1b) votes virtually disappeared.

A matrix P projects $\mathbb{R}^4$ onto a 2-dimensional subspace of $\mathbb{R}^4$.

1. What is the size of P?

(a) 4 by 4

(b) 4 by 2

(c) 2 by 2

2. What is the rank of P?

(a) 4

(b) 2

This question concerns expansion in an orthonormal basis. Depending on what has been done so far in the class, I may need to remind them that $\vec{v}_1$ and $\vec{v}_2$ are orthonormal.

It works well if used after some discussion of the advantages of orthonormal sets. Those students who know that only one dot product is needed, rather than row operations, will teach their colleagues in class discussion.

If the question is introduced before the students have enough experience using these ideas, it is better to write part (c) as

$$\vec{v}_2 \cdot \vec{x} = \frac{-29}{5} = c_2$$

in order to suggest the method.

It is important to tell the class that c_2 can also be found by row operations, with more effort.

$\vec{v}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} \frac{-4}{5} \\ \frac{3}{5} \end{bmatrix}$, and

$\vec{x} = \begin{bmatrix} 8 \\ 1 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2$. True–False:

(a) $\vec{v}_1$, $\vec{v}_2$ form an orthonormal set.

(b) Row operations are needed to find c_2.

(c) $c_2 = \frac{-29}{5}$.

This is a question about preservation of length by orthogonal matrices. The students already know that this matrix gives a rotation of $\mathbb{R}^2$.

The question generates a lot of class discussion, possibly because the last answer is so surprising. In fact, this is one of the questions that is successful because the surprising answer in part (d) encourages the students to rethink the other parts.

$$\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \ \vec{y} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \ R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

The distance between $R\vec{x}$ and $R\vec{y}$

(a) could be a long computation

(b) is the same as the distance between $\vec{x}$ and $\vec{y}$.

(c) depends on θ

(d) is 5

This question points out the difference between these two products.

I've found that many students ask about the difference between them. Perhaps the difficulty is that the dot product and the matrix are dramatically different even though the notation is only a little different.

The point of using the question as I do, is that seeing this flashed onto a screen and discussing it with your neighbor make it memorable.

$\vec{u}$ is a unit vector in $\mathbb{R}^n$. One of these

$$\vec{u}\vec{u}^T, \qquad \vec{u}^T\vec{u},$$

is a number, and one is an $n \times n$ matrix. Which is which?

Item (a) is a quick check on the Pythagorean theorem. Even though the student may have seen it stated with $\vec{u} + \vec{v}$ instead of $\vec{u} - \vec{v}$ as here, they have quickly responded True.

The students know that the $\vec{e}_k$ denote standard basis vectors. Item (b) gets few votes initially. After a minute discussing it, the class is convinced and astonished by the truth of item (b).

True–False:

(a) If $\vec{u} \cdot \vec{v} = 0$ then $||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2$

(b) In $\mathbb{R}^{50}$, $||3\vec{e}_{15} - 4\vec{e}_{39}||^2 = 25$

Here are two more questions which give practice with the dot product. Since a bit of computation is needed, we don't take an initial vote on these questions, but just let the students work on them for a minute or two. I tell them to work either with their neighbor or alone, as they prefer.

The first question is to see how these products work, while students take an active role in the middle of a big lecture. Answers: 0, 1, 0.

The second question is a good one to use when introducing orthogonal matrices. Answers: 0, $\sqrt{34}$.

The students' reactions suggest that the second question is harder than the first one. Thus the pair of questions make a good bridge to the $U^T U = I$ product for orthogonal matrices.

An orthonormal set: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. Each of the following is 0 or 1. Which is which?

$$\vec{u}_1^T \vec{u}_2, \quad \vec{u}_2^T \vec{u}_2, \quad \vec{u}_3^T \vec{u}_1^T \vec{u}_1 \vec{u}_2$$

Matrix A has three columns, $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$, and

$$A^T A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

What is $\vec{v}_1 \cdot \vec{v}_2$? $||\vec{v}_2||$?

After discussing this question briefly among themselves, the students decided that this integral version of the Pythagorean theorem was obvious, so answered (a). The idea of the question is to give a bridge to the more abstract identity at the bottom. The question is helpful because experience has shown that the identity

$$||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + 2\vec{x} \cdot \vec{y} + ||\vec{y}||^2$$

is unfamiliar even in $\mathbb{R}^3$.

Is there a missing assumption:

$$\int_a^b \big(f(t)+g(t)\big)^2\,dt =^? \int_a^b f(t)^2\,dt + \int_a^b g(t)^2\,dt$$

(a) It assumes that $\displaystyle\int_a^b f(t)g(t)\,dt = 0$.

(b) No, it is always true.

In fact, isn't $||f+g||^2 = ||f||^2 + 2\langle f, g\rangle + ||g||^2$ in any inner product space?

This question generates a lot of discussion. It is written partly to help students distinguish the current topic from the real number statement $|t| = \sqrt{t^2}$. Only (c) is certain to be true.

Suppose we are working in an inner product space of functions, containing 1 and t. True–False:

(a) $||1|| = 1$

(b) $||t|| = \sqrt{t^2}$

(c) $||t|| = \sqrt{\langle t, t\rangle}$

Chapter 6

Eigenvalues

Questions about eigenvalues.

I sometimes use this question before defining eigenvectors. That, and the large number of parts to the question, mean that students discuss it for a longer time than for most questions.

Students enjoy the question, and in particular they find humor in part (a), as I intended. They work out all the answers correctly: Humorously False, True, True, False, True.

The question taps into many common misunderstandings.

For the equation $A\vec{x} = 3\vec{x}$, which apply?

(a) We can't solve it; we only do $A\vec{x} = \vec{b}$.

(b) It is true for all A when $\vec{x}$ is zero.

(c) Is true for all $\vec{x}$ when $A = 3I$.

(d) It is not possible: cancel $\vec{x}$ to find that $A = 3$ is not a matrix at all.

(e) It implies that $(A - 3I)\vec{x} = \vec{0}$.

The question might be harder than it looks, because it attempts to remind students that, unlike in lower courses, you don't always want to dig in and solve an equation. It is enough here to know the definition of eigenvector, and try it. Sufficiently many students already know the correct approach, that the others will learn from them during their discussion.

$$B = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 7 & 0 \\ -1 & 0 & -1 \end{bmatrix}.$$

Which of these are eigenvectors of B?

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

We try to organize our knowledge. This question works well, though it initially produces some puzzled looks among the students. The idea that various statements do not have the same status or the same use is probably not something emphasized in earlier courses.

When I have asked individual students in office hours to tell me how linear dependence is defined, they may respond with any of these, or something else. Yet in the class there is, after discussion, agreement that (a) is the definition, and (b) the computational method. That is why I think this kind of question is good.

One of these statements is a definition, one is a theorem, and one is a computational method. Which is which?

(a) Vectors $v_1, ...v_r$ are linearly dependent when $\sum c_k v_k = 0$ for some c_k not all 0.

(b) You put the vectors in a matrix and reduce, and if there are not r pivots then they are dependent.

(c) If one of the vectors is 0 then the list is dependent.

Here is a question which is intended to clear up a common error. It is a very successful question. The last time I used it, the class read part (a) in silence and then suddenly burst out laughing once they read part (b). That is a good result.

I always remind students of the fundamental theorem of algebra. It seems that a few have heard of it, and a few already know that factoring and root-finding are related. This question helps the others.

Joe found out that

$$\det(A - \lambda I) = (\lambda - 6)(\lambda + 1)(\lambda - 2).$$

What should he do next?

(a) Multiply it out:

$$\begin{aligned}(\lambda - 6)(\lambda + 1)(\lambda - 2)& \\ &= (\lambda - 6)(\lambda^2 - \lambda - 2) \\ &= \lambda^3 - 7\lambda^2 - 8\lambda + 8\end{aligned}$$

then use his calculator to find the roots.

(b) Recognize that the roots are 6, -1, 2.

This question may be used after the spectral theorem for real symmetric matrices is discussed. The class has realized after some discussion that $\pm 3i$ can't be eigenvalues of such a matrix. Since no helpful hints are given, the question is somewhat difficult.

Joe says he is working on a real symmetric matrix and the characteristic polynomial is $(\lambda^2 + 3)(\lambda + 4)^2(\lambda^2 - 5)(\lambda - 6)$.
Is Joe right?

This question assumes the students know that orthogonal matrices preserve length. It points out that their eigenvalues have absolute value 1. I always mention that $|\lambda| = 1$ does not necessarily imply $\lambda = \pm 1$, since λ may be complex.

The word "orthogonal" has two meanings, as seen by comparing the following statements.

(a) A and B are orthogonal matrices.

(b) $\vec{u}$ and $\vec{v}$ are orthogonal vectors.

The meaning is entirely different. For this reason, I think it might be clearer to students at this level to say "C has orthonormal columns" rather than "C is an orthogonal matrix."

C is an orthogonal matrix and $||\vec{v}|| = 1$.

(a) What is $||C\vec{v}||$?

(b) Also $C\vec{v} = \lambda\vec{v}$. What is $||\lambda\vec{v}||$? $|\lambda|$?

This question connects ideas which might have been introduced several weeks apart, depending on the syllabus. It is worthwhile to link them.

If 0 is an eigenvalue of A, figure out the smallest possible value of $\dim(\ker A)$, and find $\det A$.

Item (b) here is taken from an error made on a test. It is fairly serious to have that kind of misunderstanding so late in the course. So I let the students discuss it. Part (c) is correct.

I have some hesitation is displaying an expression as meaningless as item (b), but have used the question in spite of this hesitation, because the students have seemed interested in it. They discuss it and decide correctly that (c) is the answer.

I tell the class that the question contains an error I found on a test paper. This might explain part of their interest.

If $A\vec{v} = \lambda\vec{v}$, what can be said about $A^2\vec{v}$?

(a) It can't be computed.

(b) It is $\lambda\vec{v}^2$.

(c) It is $\lambda^2\vec{v}$.

The class tends to puzzle over this one, usually with a positive outcome. Those who enjoy the question are prepared to see that the determinant and trace of a matrix are the product and sum of eigenvalues.

You know that

$$(x-5)(x-3) = x^2 - (3+5)x + 3 \cdot 5.$$

Similarly, if

$$\prod_{k=1}^{7}(x - r_k) = x^d - 22x^{d-1} + \cdots + 6$$

figure out d, the sum of the roots $(r_1 + \cdots + r_7)$, and the product of the roots $(r_1 r_2 \ldots r_7)$.

My students always surprise me! When I wrote this question, I wasn't sure whether item (1) would seem familiar the students. It turned out that at least thirty heads started nodding "yes" emphatically. To make good questions, ask something, see what happens, and refine it. A point connected with item (2) is that the eigenvalue system is a nonlinear system because both λ and $\vec{x}$ are unknown, and are multiplied. Possibly the nonlinearity ought to be brought out by some other question.

1. Have you ever seen it suggested in an applied text, that one needs m equations with m variables before you start solving things?
2. Think of the system $A\vec{x} = \lambda\vec{x}$, where A is 8×8. How many variables are there? equations? solutions λ? solutions $\vec{x}$?

These two questions will carry the students pretty far through the diagonalization ideas.

Depending on the text used, the first one might be the definition of matrix multiplication. The second is more interesting to the students.

These questions can be avoided by simply lecturing. But when you ask a question, the student's brain invents the mathematics for itself.

Is this correct?

$$A[\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_n] = [A\vec{v}_1\ A\vec{v}_2\ \ldots\ A\vec{v}_n]$$

Which one is true,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 5a_1 & 7b_1 & 9c_1 \\ 5a_2 & 7b_2 & 9c_2 \\ 5a_3 & 7b_3 & 9c_3 \end{bmatrix},$$

-or-

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 5a_1 & 5b_1 & 5c_1 \\ 7a_2 & 7b_2 & 7c_2 \\ 9a_3 & 9b_3 & 9c_3 \end{bmatrix}?$$

In elementary algebra a student might think of "moving" a factor to the other side of an equation, but that is not sophisticated enough to understand linear algebra. Instead you may multiply an equation by something.

In linear algebra you must decide whether to multiply on the left or on the right, and in the case shown you must think about whether the multiplier exists or not.

The class likes the question.

If $AB = BC$ then $B^{-1}AB = C$.

What did I do?

(a) Multiplied by B^{-1} on the right.

(b) Multiplied by B^{-1} on the left.

(c) Assumed B^{-1} exists.

Before using this quesion, we have talked about diagonalization of matrices and sometimes written $A = SDS^{-1}$. Depending on the text used, we've usually seen a Markov chain example in which we want to compute powers of A.

Students seem to have found the question interesting even without a reminder of those topics.

If you simplify

$$SDS^{-1}(SDS^{-1})SDS^{-1}(SDS^{-1})^4$$

as much as possible, which do you get?

(a) $SD^3(SDS^{-1})^4$

(b) SD^7S^{-1}

(c) D^7

Chapter 7

Complex scalars

Students bring some familiarity with complex numbers from high school, but a review is often needed. Most questions are about the numbers.

Thie is a question about division of complex numbers.

On a first vote, most students are unwilling to vote at all. After they discuss the question among themselves, perhaps half the class agree that all these answers are correct, and the other half still isn't voting. It is possible that this means the question is not well designed. I think it also suggests the depth of difficulty that students may have with complex numbers.

Students seem to find it interesting that while all are correct, item (d) is the version we can most easily locate on the plane.

If $(3+2i)a = i$ then a is which?

(a) not a real number

(b) $\frac{i}{3+2i}$

(c) $i\frac{3-2i}{3^2+2^2}$

(d) $\frac{2}{13} + \frac{3}{13}i$

(e) all the above

I designed these questions to find out what students know about complex numbers. I use them at first without reviewing or saying anything about the complex exponential.

Everybody agrees the first one is True. Very few students venture a response to the other two. I think this stark contrast reveals exactly where they are, what we need to work on next. It is the more striking because among these students a significant number are studying quantum mechanics too.

True–False: $e^{i\theta} = \cos(\theta) + i \sin\theta$.

The number e^i lies on the unit circle about how many degrees counterclockwise from the x axis?

(a) 180

(b) 57

(c) 22.5

(d) don't know

How far can you simplify $e^{3i}e^{0.141592653\ldots i}$?

(a) $\Big(\cos(3)+i\sin(3)\Big)\Big(\cos(0.141\ldots)+i\sin(0.141\ldots\Big)$

(b) $e^{3.141592653\ldots i}$ because $e^{a+b} = e^a e^b$

(c) $e^{i\pi} = -1$

Two common errors are to think that $|x + yi|$ is $\sqrt{x^2 + (iy)^2}$, or to forget the square root.

We write complex numbers using cartesian or polar coordinates as $z = x + iy = re^{i\theta}$, and define $|z| = r$. The magnitude $|3 - 4i|$ is which?

(a) $\sqrt{3^2 - 4^2}$

(b) 5

(c) $3^2 + 4^2$

(d) same as $|3 + 4i|$ and $| - 4 - 3i|$

The nice idea of an eigenvector as one which is merely rescaled is dashed by the 90 degree rotation.

Rotation of $\mathbb{R}^2$ by 90 degrees, sends $\vec{e}_1$ to $\vec{e}_2$ and $\vec{e}_2$ to $-\vec{e}_1$, so the matrix is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(a) Is it possible that $A\vec{x}$ is a multiple of $\vec{x}$?

(b) In $\mathbb{C}^2$ rather than $\mathbb{R}^2$, which are true?

$$A \begin{bmatrix} 1 \\ i \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ -i \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Chapter 8

Works in progress

My list of questions changes in response to conversations with students. Here are two which address important topics, and for which I don't yet have enough class feedback. They are not "good" yet.

The first question is a rough draft, while the second has been used but not refined. A third one gives an example of the LaTeXsource for a question.

You can get ideas for questions by listening to students. For example, a student recently referred to the equation of a plane, that is, the literal string of symbols, as the plane. She did not express a clear idea of a plane as a set of points satisfying the equation. It is possible that this is a common misunderstanding. These are first drafts of questions to try to find out.

Which of these statements is not accurate?

(a) A plane is a set of points.

(b) The equation

$$2x_1 - x_2 + 3x_3 = 0$$

is a plane.

How many points are there on a plane?

(a) 3

(b) infinitely many

(c) none. A plane is a linear equation.

This is an unfinished question is about the Pythagorean theorem.

I think it is important for students to develop the ability to calculate not only like

$$(x_1 + y_1)^2 + \cdots + (x_n + y_n)^2 = x_1^2 + 2x_1y_1 + y_1^2 + \cdots$$

$$= (x_1^2 + x_2^2 + \cdots) + 2(x_1y_1 + x_2y_2 + \cdots) + (y_1^2 + y_2^2 + \cdots)$$

which they can do fairly well, but also as

$$(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}$$

which seems to be harder. I plan to modify the question to probe why the second calculation is harder.

True–False:

(a) $||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$ (ever?)

(b) $||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + 2\vec{u} \cdot \vec{v} + ||\vec{v}||^2$

Questions that are good for one class will probably not be optimized for another class. As an example for typesetting your own good questions, here is the LaTeXsource for one of mine.

```
\def\boxit#1{\vbox{\hrule
                   \hbox{\vrule\kern8pt
                               \vbox{\kern4pt#1\kern8pt}
                               \kern8pt
                         \vrule}
                   \hrule}}


\begin{huge}
\boxit{
$$
 A = \begin{bmatrix} 1 & 2 & 0 \\
                     3 & 0 & 6 \\
                    -2 & 1 & 0 \end{bmatrix},
 \quad
 \vec{v} = \begin{bmatrix} 0\\ 0\\ \frac{1}{6} \end{bmatrix},
 \quad
  A^{-1} {\rm \ exists}.
$$
\begin{enumerate}
 \item True-False: $A\vec{v}=\vec{e}_2$
 \item True-False: $A^{-1}\vec{e}_2 = \vec{v}$
 \item What is column 2 of $A^{-1}$?
\end{enumerate}
}
\end{huge}
```

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 6 \\ -2 & 1 & 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{6} \end{bmatrix}, \quad A^{-1} \text{ exists.}$$

(a) True-False: $A\vec{v} = \vec{e}_2$

(b) True-False: $A^{-1}\vec{e}_2 = \vec{v}$

(c) What is column 2 of A^{-1}?

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